

THE MATROID SECRETARY PROBLEM FOR MINOR-CLOSED CLASSES AND RANDOM MATROIDS

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ABSTRACT. We prove that for every proper minor-closed class \mathcal{M} of \mathbb{F}_p -representable matroids, there exists a $O(1)$ -competitive algorithm for the matroid secretary problem on \mathcal{M} . This result relies on the extremely powerful matroid minor structure theory being developed by Geelen, Gerards and Whittle.

We also note that for asymptotically almost all matroids, the matroid secretary algorithm that selects a random basis, ignoring weights, is $(2 + o(1))$ -competitive. In fact, assuming the conjecture that almost all matroids are paving, there is a $(1 + o(1))$ -competitive algorithm for almost all matroids.

1. INTRODUCTION

The *matroid secretary problem* was introduced by Babaioff, Immorlica, and Kleinberg [1] as a generalization of the classical secretary problem. The setup is as follows. We are given a matroid M whose elements are ‘secretaries’ with an (unknown) weight function $w: E(M) \rightarrow \mathbb{R}_{\geq 0}$. The secretaries are presented to us online in a random order. When a secretary e is presented to us, we learn its weight $w(e)$. At this point, we must make an irrevocable decision to either hire e or not. The *weight* of a set of secretaries is the sum of their weights. Our goal is to design an algorithm that hires a set of secretaries with large weight, subject to the constraint that the hired set of secretaries are an independent set in M . The *value* of an algorithm on (M, w) is the average weight of an independent set it produces, taken over all orderings of $E(M)$. We write $\text{OPT}(M, w)$ for the maximum weight of an independent set of M . For $c \geq 1$, an algorithm is c -competitive for M if the value it outputs for (M, w) is at least $\frac{1}{c} \text{OPT}(M, w)$ for all weight functions $w: E(M) \rightarrow \mathbb{R}_{\geq 0}$. The following deep conjecture remains open.

Conjecture 1 (Babaioff, Immorlica, and Kleinberg [1]). *For all matroids M , there is a $O(1)$ -competitive matroid secretary algorithm for M .*

Babaioff, Immorlica, and Kleinberg [1] gave a $O(\log r)$ -competitive algorithm for all matroids M , where r is the rank of M . This was improved to a $O(\sqrt{\log r})$ -competitive algorithm by Chakraborty and Lachish [2]. The state-of-the-art is a $O(\log \log r)$ -competitive algorithm, first obtained by Lachish [17]. Using different tools, Feldman, Svensson, and Zenklusen [6] give a simpler $O(\log \log r)$ -competitive algorithm for all matroids.

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On the other hand, there are constant-competitive algorithms for restricted classes of matroids. Graphic matroids have a $2e$ -competitive algorithm due to Korula and Pál [16]. Cographic matroids have a $3e$ -competitive algorithm due to Soto [24]. Regular matroids and max-flow min-cut matroids both have $9e$ -competitive algorithms due to Dinitz and Kortsarz [4].

Our first result is a vast generalization of all the aforementioned constant-competitive algorithms. To be forthright with the reader, we stress that this result relies on a deep structure theorem communicated to us by Geelen, Gerards and Whittle, which has not yet appeared in print. Therefore, we state this structure theorem as Hypothesis 17. Roughly, Hypothesis 17 asserts that every matroid in a proper minor-closed class of \mathbb{F}_p -representable matroids admits a tree-like decomposition into sparse or graph-like pieces. The proof of (a stronger version of) Hypothesis 17 will stretch to hundreds of pages and will be a consequence of their decade-plus ‘matroid minors project’. This is a body of work generalising Robertson and Seymour’s graph minors structure theorem [22] to matroids representable over a fixed finite field, leading to a solution of Rota’s Conjecture [9]. See [10] for a discussion of the project.

A class of matroids \mathcal{M} is *minor-closed* if $M \in \mathcal{M}$ and N a minor of M implies that N is also in \mathcal{M} . A *proper minor-closed* class of \mathbb{F}_p -representable matroids is a minor-closed class of \mathbb{F}_p -representable matroids that is not equal to the class of all \mathbb{F}_p -representable matroids.

Theorem 2. *Suppose that Hypothesis 17 holds. Let p be a prime and \mathcal{M} be a proper minor-closed class of the \mathbb{F}_p -representable matroids. Every matroid in \mathcal{M} has an $O(1)$ -competitive matroid secretary algorithm.*

Note that the existence of constant-competitive algorithms for transversal matroids [3] and laminar matroids [13, 12, 14] is not implied by Theorem 2. In addition, the matroid secretary algorithm in Theorem 2 requires knowledge of the entire matroid upfront. This is the same model as originally introduced by Babaioff, Immorlica, and Kleinberg [1], but many of the known matroid secretary algorithms also work in the online setting, see [11]. We do not know if Theorem 2 holds in the online setting.

Our proof of Theorem 2 extends the nice framework of Dinitz and Kortsarz [4] for obtaining constant-competitive algorithms for ‘decomposable’ matroids. In Theorem 13, we prove that if M admits a certain type of ‘tree-decomposition’ into matroids for which we already have constant-competitive algorithms, then there is a constant-competitive algorithm for M itself. Note that the proof of Theorem 13 is independent of any matroid structure theory results. For example, plugging in the regular matroid decomposition theorem of Seymour [23] into Theorem 13, we recover the constant competitive algorithm for regular matroids. Our framework is slightly more general than [4], where only 1-, 2-, and 3-sums of matroids are considered, while we allow tree-decompositions of any fixed ‘thickness’. Plugging in the matroid structure machinery into Theorem 13 gives Theorem 2.

Our second result shows that Conjecture 1 holds for almost all matroids. For each M , let RB denote the matroid secretary algorithm that chooses a uniformly random basis B of M , and selects precisely the elements of B , ignoring weights.

Theorem 3. *For asymptotically almost all matroids on n elements, the algorithm RB is $(2 + \frac{3}{\sqrt{n}})$ -competitive.*

The veracity of Theorem 3 is not terribly surprising, although given the limited tools in asymptotic matroid theory, it is a bit surprising that we can prove it. Indeed, the proof of Theorem 3 relies on recent breakthrough results of Pendavingh and van der Pol [20, 21].

One can think of Theorem 2 and Theorem 3 as complementary results. Theorem 3 asserts that the ‘typical’ matroid has a constant-competitive algorithm, so we expect counterexamples to Conjecture 1 to come from structured classes of matroids. On the other hand, Theorem 2 says that restricting to proper-minor closed classes of \mathbb{F}_p -representable matroids also cannot produce counterexamples to Conjecture 1.

Recall that for the classical secretary problem, the algorithm that samples and rejects the first $\frac{1}{e}$ of the secretaries and then hires the first secretary that is better than all secretaries in the sample is e -competitive. Dynkin [5] showed that this algorithm is best possible. Thus, interestingly, for almost all matroids the algorithm RB has a better competitive ratio than the best classical secretary algorithm. Indeed, under a widely believed conjecture in asymptotic matroid theory, we note that there is in fact a $(1 + o(1))$ -competitive algorithm for almost all matroids.

Theorem 4. *If asymptotically almost all matroids are paving, then there is a $(1 + o(1))$ -competitive matroid secretary algorithm for asymptotically almost all matroids.*

We will discuss and prove Theorem 3 and 4 in the last section of the paper.

2. PRELIMINARIES

In this section, we review some results that we will require later. We assume basic knowledge of matroid theory and follow the notation of Oxley [19].

We begin by quickly defining the matroids that appear in this paper. A matroid is *graphic* if it is isomorphic to the cycle matroid of a graph. It is *cographic* if it is the dual of a graphic matroid. Let \mathbb{F} be a field. A matroid is \mathbb{F} -*representable* if it is isomorphic to the column matroid of a matrix with entries in \mathbb{F} , and it is *regular* if it is \mathbb{F} -representable for every field \mathbb{F} . A *represented frame matroid* is a matroid that has a matrix representation in which each column has at most two nonzero entries. Finally, we let $M(K_n)$ denote the cycle matroid of the complete graph K_n and $U_{r,n}$ denote the matroid on $\{1, \dots, n\}$ where all r -subsets are bases. The latter matroids are called *uniform matroids*.

Let M be a matroid with ground set E and let C and D be disjoint subsets of E . We let $M/C \setminus D$ denote the matroid obtained from M by contracting the elements in C and deleting the elements in D . A matroid N is a *minor* of a matroid M if N is isomorphic to $M/C \setminus D$ for some choice of C and D . We let $\text{si}(M)$ be the matroid obtained from M by suppressing loops and parallel elements.

We will use two theorems of Soto. The first ([24], Theorem 5.2) gives a constant-competitive algorithm for ‘sparse’ matroids.

Theorem 5. *Let $\gamma \in \mathbb{R}$. If M is a matroid so that $N = \text{si}(M)$ satisfies $|X| \leq \gamma r_N(X)$ for all $X \subseteq E(N)$, then there is a γe -competitive matroid secretary algorithm for M .*

The next ([24], Theorem 5.4) gives a constant-competitive algorithm for representable frame matroids.

Theorem 6. *If M is a represented frame matroid, then there is a $2e$ -competitive matroid secretary algorithm for M .*

Finally, we use a trivial lemma that is proved by setting elements to have zero weight appropriately.

Lemma 7. *If there is a c -competitive matroid secretary algorithm for M , then there is a c -competitive algorithm for every restriction of M .*

3. LIFTING AND PROJECTION

Let M_1 and M_2 be matroids on a common ground set E . We say that M_1 is a *distance-1 perturbation* of M_2 if there is a matroid M and a nonloop element x of M such that $\{M/x, M \setminus x\} = \{M_1, M_2\}$. We say that M/x is a *projection* of $M \setminus x$ and $M \setminus x$ is a *lift* of M/x (many authors call these *elementary* projections/lifts). The *perturbation distance* between two matroids M, M' on a common ground set is the minimum t for which there exists a sequence $M = M_0, M_1, \dots, M_t = M'$ where each M_i is a distance-1 perturbation of M_{i-1} . We write $\text{dist}(M, M')$ for this quantity. Note that if M and M' are representable (say by matrices A and A'), then $\text{dist}(M, M')$ is small if and only if $A' = A + P$ for some low-rank matrix P .

In this section, we show that the existence of constant-competitive algorithms is robust under a bounded number of lifts/projections.

Lemma 8. *Let N be a lift of a matroid M . If there is a c -competitive algorithm for M then there is a $\max(e, 2c)$ -competitive algorithm for N .*

Proof. Let ALG_M be a c -competitive algorithm for M . Let L be a matroid and x be a nonloop of L such that $L/x = M$ and $L \setminus x = N$. Let P be the parallel class of L containing x . Note that each element in $P - \{x\}$ is a loop in M , and hence will never be selected by ALG_M . We specify an algorithm ALG_N for N as follows:

- as the elements of $P - \{x\}$ are received, ALG_N runs the classical secretary algorithm to select one. If $P = \{x\}$, then no element is chosen in this way.
- as the elements of $E(M) - P$ are received, ALG_N passes them to ALG_M and selects them as ALG_M does.

Let I be the set of elements chosen in the first way (so $|I| \leq 1$) and J be the set of elements chosen in the second way. Clearly J is independent in N/I and so $I \cup J$ is independent in N . It remains to show that $\mathbf{E}(w(I \cup J)) \geq \frac{1}{\max(e, 2c)} \text{OPT}(N, w)$. Let B be a max-weight basis of (N, w) and let C be the unique circuit of L with $\{x\} \subseteq C \subseteq B \cup \{x\}$. To analyse ALG_N we distinguish two cases.

If $|C| \geq 3$, then let $C' = C - \{x\}$ and let y be a minimum-weight element of C' . Now $w(C' - \{y\}) \geq \frac{1}{2}w(C')$ and $B - \{y\}$ is a basis of M satisfying $w(B - \{y\}) = w(B - C') + w(C' - \{y\}) \geq w(B - C') + \frac{1}{2}w(C') \geq \frac{1}{2}w(B)$. Therefore $\text{OPT}(M, w) \geq \frac{1}{2} \text{OPT}(N, w)$ and so $\mathbf{E}(w(J)) \geq \frac{1}{c} \text{OPT}(M, w) \geq \frac{1}{2c} \text{OPT}(N, w)$.

If $|C| < 3$, then (since x is a nonloop of L) we have $C = \{x, x'\}$ for some $x' \in P - \{x\}$. In this case $B = \{x'\} \cup J'$ for some independent set J' of M . Now I is chosen by an e -competitive secretary algorithm on $P - \{x\}$, so $\mathbf{E}(w(I)) \geq \frac{1}{e}w(x')$. Moreover, since

B is optimal and $\{x'\} \cup B_0$ is independent in N for every basis B_0 of M , we have $w(B - \{x'\}) = \text{OPT}(M, w)$. Therefore

$$\begin{aligned} \mathbf{E}(w(I \cup J)) &\geq \frac{1}{e}w(x') + \frac{1}{c}\text{OPT}(M, w) \\ &= \frac{1}{e}w(x') + \frac{1}{c}w(B - \{x'\}) \\ &\geq \frac{1}{\max(e, c)}\text{OPT}(N, w). \end{aligned}$$

It follows from these two cases that ALG_N is $\max(e, 2c)$ -competitive for N . \square

Lemma 9. *Let N be a projection of a matroid M . Let L be the set of loops of N . If there is a c -competitive algorithm for M then there is an $(e+1)c$ -competitive algorithm for $M \setminus L$ whose output is always independent in N .*

Proof. By Lemma 7, there is a c -competitive algorithm $\text{ALG}_{M \setminus L}$ for $M \setminus L$. Let P be a matroid and x be a nonloop of P so that $P/x = N$ and $P \setminus x = M$. Let X be the random variable taking the value **heads** with probability $\frac{e}{e+1}$ and **tails** with probability $\frac{1}{e+1}$. We define another matroid secretary algorithm ALG_N for $M \setminus L$ as follows:

- if $X = \text{heads}$, then we run a classical secretary algorithm on $N \setminus L$ to select just one element.
- if $X = \text{tails}$, then we run $\text{ALG}_{M \setminus L}$, except we only pretend to hire any element whose selection would create a dependency in N with the elements already chosen.

Fix a weighting w of $M \setminus L$. Let $x_0 \in E(M \setminus L)$ have maximum weight, and let I be the set selected by ALG_N . Clearly if $X = \text{heads}$ then I is independent in N , and we have $\mathbf{E}(w(I) | X = \text{heads}) \geq \frac{1}{e}w(x_0)$. Moreover, if J is the set output by $\text{ALG}_{M \setminus L}$, then $J \cup \{x\}$ contains at most one circuit of P . It follows that if $X = \text{tails}$ then I is obtained from J by removing at most one element, so $w(I) \geq w(J) - w(x_0)$. Thus

$$\begin{aligned} \mathbf{E}(w(I)) &= \frac{e}{e+1}\mathbf{E}(w(I) | X = \text{heads}) + \frac{1}{e+1}\mathbf{E}(w(I) | X = \text{tails}) \\ &\geq \frac{1}{e+1}w(x_0) + \frac{1}{e+1}(\mathbf{E}(w(J) - w(x_0)) | X = \text{tails}) \\ &= \frac{1}{e+1}\mathbf{E}(w(J) | X = \text{tails}) \\ &\geq \frac{1}{c(e+1)}\text{OPT}(M \setminus L, w). \end{aligned}$$

It follows that ALG_N is $(e+1)c$ -competitive for $M \setminus L$. \square

In particular, since every independent set in N is an independent set of $M \setminus L$, the algorithm ALG_N is $(e+1)c$ -competitive for N . Since $(e+1)c > \max(e, 2c)$ for $c \geq 1$, we can thus combine Lemmas 8 and 9 with an inductive argument to yield the following.

Lemma 10. *Let $t \in \mathbb{N}$ and let M and N be matroids with $\text{dist}(M, N) \leq t$. If there is a c -competitive algorithm for M , then there is a $(e+1)^t c$ -competitive algorithm for N .*

Similarly, iterating Lemma 9 yields the following:

Lemma 11. *Let N be a matroid obtained from a matroid M by a sequence of t projections. Let L be the set of loops of N . If there is a c -competitive algorithm for M , then there is a $c(e+1)^t$ -competitive algorithm for $M \setminus L$ whose output is always independent in N .*

4. TREE-DECOMPOSITIONS

In this section, we introduce a notion of tree-decompositions of matroids, and give a constant-competitive matroid secretary algorithm for matroids having a ‘bounded-thickness’ tree-decomposition into pieces for which constant-competitive algorithms are known. We use the term *thickness*, to distinguish our notion from other well-known width parameters for matroids such as *branch-width* [7].

For a matroid $M = (E, r)$ and disjoint sets $X, Y \subseteq E$, we write $\sqcap_M(X, Y) = r_M(X) + r_M(Y) - r_M(X \cup Y)$. The *connectivity function* λ_M of a matroid M is the function $\lambda_M: 2^E \rightarrow \mathbb{Z}_{\geq 0}$ defined by $\lambda_M(X) = \sqcap_M(X, E - X)$. Note that if M is graphic, then λ_M (essentially) encodes graph connectivity (see Oxley [19]). It is also easy to check that λ_M is a symmetric and submodular function. We use the following easy lemma:

Lemma 12. *Let $M = (E, r)$ be a matroid and let $X \subseteq E$. Then $M/(E - X)$ is obtained from $M|X$ by a sequence of $\lambda_M(X)$ projections.*

Proof. Let $I_1 \subseteq X$ and $I_2, I_3 \subseteq E - X$ be disjoint independent sets of M so that I_1 is a basis for $M|X$, $I_1 \cup I_2$ is a basis for M , and $I_2 \cup I_3$ is a basis for $M \setminus X$. We have $|I_3| = r_M(E - X) - (r_M(E) - r_M(X)) = \lambda_M(X)$. Now the matroid $N = (M/I_2)|(X \cup I_3)$ satisfies $N/I_3 = M/(E - X)$ and $N \setminus I_3 = M|X$. The lemma follows. \square

A *tree-decomposition* of a matroid M is a pair (T, \mathcal{X}) where T is a tree and $\mathcal{X} := \{X_v : v \in V(T)\}$ is a partition of $E(M)$ indexed by $V(T)$. Let $e = v_1 v_2 \in E(T)$ and T_1 and T_2 be the components of $T \setminus e$ where $v_i \in V(T_i)$. Let $X_1 := \bigcup_{v \in V(T_1)} X_v$. We define the *thickness* of e , $\lambda(e)$, to be $\lambda_M(X_1)$. The *thickness* of (T, \mathcal{X}) is $\max_{e \in E(T)} \lambda(e)$. Given $v \in V(T)$, we write $M(v)$ for $M|X_v$; this is a restriction of M . If, for all $e = uv \in E(T)$, we have $\sqcap_M(X_u, X_v) = \lambda(e)$, then we say (T, \mathcal{X}) is a *full tree-decomposition* of M .

Theorem 13. *Let \mathcal{M} be a class of matroids for which there exists a c -competitive matroid secretary algorithm. Let $k \in \mathbb{N}$ and let $t_k(\mathcal{M})$ be the set of all matroids M with a full tree-decomposition (T, \mathcal{X}) of thickness at most k such that $M|cl_M(X_v) \in \mathcal{M}$ for each $v \in V(T)$. Then there is an $c(e + 1)^k$ -competitive matroid secretary algorithm for $t_k(\mathcal{M})$.*

Proof. We say a tree-decomposition (T, \mathcal{X}) of a matroid M is an \mathcal{M} -tree decomposition if $M|cl_M(X_v) \in \mathcal{M}$ for all $v \in V(T)$. For each $m \geq 1$, let $t_{k,m}(\mathcal{M})$ denote the class of matroids in $t_k(\mathcal{M})$ having a full \mathcal{M} -tree-decomposition (T, \mathcal{X}) of thickness at most k with $|V(T)| \leq m$. There is clearly a $c(e + 1)^k$ -competitive matroid secretary algorithm for every matroid in $t_{k,1}(\mathcal{M}) = \mathcal{M}$. Let $m > 1$ and suppose inductively that every matroid in $t_{k,m-1}(\mathcal{M})$ has a $c(e + 1)^k$ -competitive matroid secretary algorithm.

Let $M \in t_{k,m}(\mathcal{M})$, let $E = E(M)$, and let (T, \mathcal{X}) be a full \mathcal{M} -tree-decomposition of M of thickness at most k with $1 < |V(T)| \leq m$. Let ℓ be a leaf of T and let $e = \ell u$ be the edge of T incident with ℓ . Let X'_ℓ and X'_u be obtained from X_ℓ and X_u by moving all elements from $X_\ell \cap cl_M(X_u)$ into X_u . It is easy to check that $\lambda_M(X'_\ell) \leq \lambda_M(X_\ell)$ and that $\lambda_M(X'_\ell) = \sqcap_M(X'_\ell, X'_u)$. Furthermore, since $cl_M(X'_u) = cl_M(X_u)$, it follows that $\sqcap_M(X'_u, X_v) = \sqcap_M(X_u, X_v)$ for all $v \notin \{u, \ell\}$. Therefore, this move preserves the property of being a full \mathcal{M} -tree-decomposition of thickness at most k ; and so we may assume that $X_\ell \cap cl_M(X_u) = \emptyset$.

Let $M(\ell) = M|X_\ell$ and let $M'(\ell) = M/(E - X_\ell)$. By Lemma 12, the latter is obtained from the former by at most $\lambda_M(X_\ell) \leq k$ projections. Moreover, since $\sqcap_M(X_\ell, X_u) = \lambda_M(X_\ell)$, we have $\lambda_{M/X_u}(X_\ell) = 0$ and so $M'(\ell) = (M/X_u)|X_\ell$; since $X_\ell \cap \text{cl}_M(X_u) = \emptyset$ it follows that $M'(\ell)$ has no loops.

By Lemmas 7 and 11, there is a $c(e+1)^k$ -competitive algorithm ALG_ℓ for $M(\ell)$ whose output is always independent in $M'(\ell)$. Moreover, we see that $(T \setminus \ell, \{X_w : w \in V(T \setminus \ell)\})$ is a full \mathcal{M} -tree-decomposition of $M \setminus X_\ell$ with thickness at most k , so $M \setminus X_\ell \in t_{k,m-1}(\mathcal{M})$ and there is thus a $c(e+1)^k$ -competitive algorithm ALG' for $M \setminus X_\ell$. Define an algorithm ALG for M by running ALG' and ALG_ℓ on the elements of $E - X_\ell$ and X_ℓ respectively as they are received, choosing all elements chosen by either.

Let I_ℓ and I' be the sets chosen by ALG_ℓ and ALG' respectively. Since I' is independent in M and I_ℓ is independent in $M'(\ell)$, the set $I = I_\ell \cup I'$ chosen by ALG is independent in M . Moreover, for each weighting w of M we have

$$\begin{aligned} \mathbf{E}(w(I)) &= \mathbf{E}(w(I')) + \mathbf{E}(w(I_\ell)) \\ &\geq \frac{1}{c(e+1)^k} (\text{OPT}(M(\ell), w|X_\ell) + \text{OPT}(M \setminus X_\ell, w|(E - X_\ell))) \\ &\geq \frac{1}{c(e+1)^k} \text{OPT}(M, w), \end{aligned}$$

since each independent set of M is the union of an independent set of $M(\ell)$ and one of $M \setminus X_\ell$. The theorem follows. \square

As an easy corollary of Theorem 13, we obtain a short proof that there is an $O(1)$ -competitive algorithm for regular matroids, a result first proved by Dinitz and Kortsarz [4]. The constant $9e$ that they obtain is better than ours by a factor of $\frac{1}{3}(e+1)^2 \approx 4.6$.

Corollary 14. *There is a $3e(e+1)^2$ -competitive matroid secretary algorithm for each regular matroid.*

Proof. By Seymour's regular matroid decomposition theorem [23], every regular matroid M is obtained from pieces that are either graphic, cographic or R_{10} by 1-, 2- or 3-sums. This gives a tree-decomposition (T, \mathcal{X}) of thickness at most 2 in M so that each $M|X_v$ is either graphic, cographic or R_{10} . Moreover, by performing parallel extensions of the elements to be deleted before each 2-sum and 3-sum, one can construct a matroid M' having M as a restriction and a *full* tree-decomposition (T, \mathcal{X}') of M' so that each $M'| \text{cl}_{M'}(X'_v)$ is either graphic, cographic or a parallel extension of R_{10} .

Korula and Pál [16] proved that there is a $2e$ -competitive matroid secretary algorithm for graphic matroids. Soto [24] proved that there is a $3e$ -competitive matroid secretary algorithm for cographic matroids. Since R_{10} is the union of two bases, Theorem 5 implies that each of its parallel extensions has a $2e$ -competitive algorithm. It follows from Theorem 13 that there is a $3e(e+1)^2$ -competitive algorithm for M' ; by Lemma 7 there is also one for M . \square

5. PROPER MINOR-CLOSED CLASSES

In this section we show that, contingent on a certain deep structure theorem, there is a constant-competitive algorithm for every proper minor-closed subclass of the matroids

representable over a fixed prime field. As well as this structure theorem, we require one other result, due to Geelen ([8], Theorem 4.3).

Theorem 15. *Let $q, n \in \mathbb{N}$. If M is a simple matroid with no $U_{2,q+2}$ -minor and no $M(K_n)$ -minor, then $|M| \leq q^{q^{3n}} r(M)$.*

Since no restriction of such an M has either of the forbidden minors, for such an M we also have $|X| \leq q^{q^{3n}} r_M(X)$ for each $X \subseteq E(M)$. Combining this theorem with Theorem 6, we thus have the following.

Corollary 16. *Let $q, n \in \mathbb{N}$. If M is a matroid with no $U_{2,q+2}$ -minor and no $M(K_n)$ -minor, then there is a $eq^{q^{3n}}$ -competitive matroid secretary algorithm for M .*

We now state the structure theorem we will use to prove Theorem 2.

Hypothesis 17. *Let p be a prime. For every proper minor-closed class \mathcal{M} of \mathbb{F}_p -representable matroids, there exist k, n and t for which every $M \in \mathcal{M}$ is a restriction of a matroid M' having a full tree-decomposition (T, \mathcal{X}) of thickness at most k such that for all $v \in V(T)$, if $M'|_{\text{cl}_{M'}(X_v)}$ has an $M(K_n)$ -minor, then there is a represented frame matroid N with $\text{dist}(M'|_{\text{cl}_{M'}(X_v)}, N) \leq t$.*

We can now prove Theorem 2, which we restate here.

Theorem 2. *Suppose that Hypothesis 17 holds. If p is prime and \mathcal{M} is a proper minor-closed subclass of the \mathbb{F}_p -representable matroids, then there exists $c = c(\mathcal{M})$ so that every $M \in \mathcal{M}$ has a c -competitive matroid secretary algorithm.*

Proof. Let k, n and t be the integers given for \mathcal{M} by Hypothesis 17. Let \mathcal{M}_1 denote the class of matroids having perturbation distance at most t from some represented frame matroid. Let \mathcal{M}_2 denote the class of matroids with no $U_{2,p+2}$ -minor or $M(K_n)$ -minor. By Theorem 6 and Lemma 10, every matroid in \mathcal{M}_1 has a $2e(e+1)^t$ -competitive matroid secretary algorithm. Corollary 16 gives a $ep^{p^{3n}}$ -competitive matroid secretary algorithm for each matroid in \mathcal{M}_2 . It is also easy to show that $U_{2,p+2}$ is not \mathbb{F}_p -representable. Thus, by Hypothesis 17 every $M \in \mathcal{M}$ is a restriction of a matroid M' having a proper tree-decomposition (T, \mathcal{X}) of thickness at most k so that $M|_{\text{cl}_{M'}(X_v)} \in \mathcal{M}_1 \cup \mathcal{M}_2$ for each $v \in V(T)$. By Theorem 13, every such M' has a c -competitive matroid secretary algorithm, where $c = (e+1)^k \max(2e(e+1)^t, ep^{p^{3n}})$. By Lemma 7, the same is true for every $M \in \mathcal{M}$. \square

6. ASYMPTOTIC RESULTS

We say *asymptotically almost all matroids* have a property if the proportion of matroids with ground set $\{1, \dots, n\}$ having the property tends to 1 as n approaches infinity. We finish our paper by showing that asymptotically almost all matroids have a constant-competitive matroid secretary algorithm. We require two recent results of Pendavingh and van der Pol [20, 21].

Theorem 18. *There exists $\alpha > 0$ so that asymptotically almost all matroids on n elements have at least $\left(1 - \frac{\alpha(\log n)^3}{n}\right) \binom{n}{r}$ bases.*

Theorem 19. *If $\beta > \sqrt{\frac{1}{2} \ln(2)}$, then asymptotically almost all matroids on n elements have rank between $\frac{n}{2} - \beta\sqrt{n}$ and $\frac{n}{2} + \beta\sqrt{n}$.*

Recall that RB denotes the algorithm that selects a basis B uniformly at random from the set of all bases of M , and chooses the elements of B as secretaries regardless of the weights. By Theorem 18, nearly every r -set in a typical matroid is a basis, so a uniformly random basis can be sampled in probabilistic polynomial time in almost all matroids by repeatedly choosing a uniformly random r -set until a basis is chosen.

We now state and prove a stronger version of Theorem 3.

Theorem 3. *Let $\gamma > \sqrt{8 \ln(2)}$. For asymptotically almost all matroids M on n elements, the algorithm RB is $(2 + \frac{\gamma}{\sqrt{n}})$ -competitive for M .*

Proof. Let $\gamma' \in (\sqrt{8 \ln(2)}, \gamma)$. Note that for each $\alpha \in \mathbb{R}$ we have $\frac{1}{2} - \frac{\gamma'}{4\sqrt{n}} - \frac{\alpha(\log n)^3}{n} > (2 + \frac{\gamma}{\sqrt{n}})^{-1}$ for all sufficiently large n . Let n be a positive integer and let M be a matroid on n elements with ground set $E = \{1, \dots, n\}$ and rank r . Let $\mathcal{B} \subseteq \binom{E}{r}$ denote the set of bases of M . For each $e \in E$ let $\mathcal{B}_e = \{B \in \mathcal{B} : e \in B\}$. By Theorems 18 and 19, there exists $\alpha \in \mathbb{R}$ so that, asymptotically almost surely, $|\mathcal{B}| \geq (1 - \frac{\alpha(\log n)^3}{n})\binom{n}{r}$ and $r \geq (\frac{1}{2} - \frac{\gamma'}{4\sqrt{n}})n$. Thus, almost surely,

$$\begin{aligned} |\mathcal{B}_e| &\geq |\{B \in \binom{E}{r} : e \in B\}| - \left(\binom{n}{r} - |\mathcal{B}| \right) \\ &= \frac{r}{n} \binom{n}{r} - \binom{n}{r} + |\mathcal{B}| \\ &= \binom{n}{r} \left(\frac{|\mathcal{B}|}{\binom{n}{r}} + \frac{r}{n} - 1 \right) \\ &\geq |\mathcal{B}| \left(1 - \frac{\alpha(\log n)^3}{n} + \frac{1}{2} - \frac{\gamma'}{4\sqrt{n}} - 1 \right) \\ &\geq |\mathcal{B}| (2 + \frac{\gamma}{\sqrt{n}})^{-1}. \end{aligned}$$

Let B_0 be a basis chosen uniformly at random from \mathcal{B} , as per RB. For each $e \in E$, we have $\mathbf{P}(e \in B_0) = |\mathcal{B}_e|/|\mathcal{B}| \geq (2 + \frac{\gamma}{\sqrt{n}})^{-1}$. Given a weighting w of M , we therefore have

$$\mathbf{E}(w(B_0)) = \sum_{e \in E} w(e) \mathbf{P}(e \in B_0) \geq (2 + \frac{\gamma}{\sqrt{n}})^{-1} w(E) \geq (2 + \frac{\gamma}{\sqrt{n}})^{-1} \text{OPT}(M, w).$$

RB is thus $(2 + \frac{\gamma}{\sqrt{n}})$ -competitive. \square

A rank- r matroid is *paving* if all its circuits have cardinality at least r . Although few well-known constructions of matroids have this property, it is believed to almost always hold. The following conjecture due to Mayhew, Newman, Welsh, and Whittle [18] is central in asymptotic matroid theory.

Conjecture 20. *Asymptotically almost all matroids on n elements are paving.*

An alternative characterisation is that a matroid M is paving if and only if its truncation to rank $r(M) - 1$ (which we denote as $T(M)$) is a uniform matroid. The matroid secretary problem for uniform matroids is known as the *multiple-choice*

secretary problem, and has been essentially completely solved by Kleinberg [15]. In the language of matroids, his result is the following.

Theorem 21. *Let $r > 25$. There is an algorithm **UNI** that is $(1 - \frac{5}{\sqrt{r}})^{-1}$ -competitive for each uniform matroid $U_{r,n}$.*

We slightly modify **UNI** to deal with paving matroids. Let **PAV** be the algorithm that, given a rank- $(r > 26)$ weighted paving matroid (M, w) , returns the output of **UNI** on the rank- $(r - 1)$ weighted uniform matroid $(T(M), w)$. Note that, since every independent set in $T(M)$ is independent in M , this output is always legal.

Theorem 22. *The algorithm **PAV** is $(1 - \frac{6}{\sqrt{r}})^{-1}$ -competitive for every paving matroid of rank at least 37.*

Proof. Let B be a maximum-weight basis of a rank- r paving matroid M with weights w , and let $x \in B$ have minimum weight. Now $B - \{x\}$ is a basis of $T(M)$ and has weight at least $\frac{r-1}{r}w(B)$, so $\text{OPT}(T(M), w) \geq (1 - \frac{1}{r})\text{OPT}(M, w)$. Let I be the independent set output by **UNI** on $T(M)$. Now

$$\mathbf{E}(w(I)) \geq (1 - \frac{5}{\sqrt{r-1}})\text{OPT}(T(M), w) \geq (1 - \frac{5}{\sqrt{r-1}})(1 - \frac{1}{r})\text{OPT}(M, w).$$

Since $(1 - \frac{5}{\sqrt{r-1}})(1 - \frac{1}{r}) \geq 1 - \frac{6}{\sqrt{r}}$ for all $r \geq 37$, **PAV** is thus $(1 - \frac{6}{\sqrt{r}})^{-1}$ -competitive. \square

As promised, we finish with a proof of Theorem 4, which we restate for convenience.

Theorem 4. *If asymptotically almost all matroids are paving, then there is a $(1 + o(1))$ -competitive matroid secretary algorithm for asymptotically almost all matroids.*

Proof. By Theorem 19, almost all matroids have $37 < r \approx \frac{n}{2}$ and so, conditional on Conjecture 20, Theorem 22 implies that **PAV** is $(1 + \epsilon(n))$ -competitive for asymptotically almost all matroids on n elements, where $\epsilon(n) \approx \frac{6\sqrt{2}}{\sqrt{n}}$. \square

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